A RANK $(2g-1)$ AFFINE-PRYM CONSTRUCTION AND ITS SCALAR TWO-BLOCK OPTIMALITY

Abstract. Let Σ_g be a closed connected oriented surface of genus $g \geq 2$, and let

$$\psi : \pi_1(\Sigma_g) \longrightarrow \mathbb{C}^\times$$

be a non-trivial finite scalar character. Put

$$V_\psi := H^1(\Sigma_g; \mathbb{C}_\psi), \qquad \dim_{\mathbb{C}} V_\psi = 2g-2.$$

We construct an explicit rank $2g-1$ non-semisimple representation of $\pi_1(\Sigma_g)$ with infinite image and finite ordinary mapping-class-group orbit. The construction is the universal affine-Prym extension attached to the full Prym cohomology space V_ψ. For punctured surfaces, the same construction applies to $W_1 H^1(\Sigma_{g,n}; \mathbb{C}_\psi) \cong H^1(\Sigma_g; \mathbb{C}_\psi)$ under peripheral triviality.

The second result proves that this rank is optimal inside the restricted class of scalar two-block triangular Prym extensions

$$0 \longrightarrow \mathbb{C}_\psi \otimes M \longrightarrow \rho \longrightarrow \mathbb{C} \otimes N \longrightarrow 0.$$

Namely, if such an extension has infinite image and finite ordinary mapping-class-group orbit, then

$$\dim M + \dim N \geq \dim V_\psi + 1 = 2g-1.$$

The proof combines Looijenga's cyclic Prym image theorem, the standard Borel-density passage for the associated arithmetic unitary groups, and Westwick's bound for complex linear spaces of matrices of fixed rank. The resulting Prym monodromy statement is a single-character projective closure: the connected projective Zariski closure of the stabilizer of ψ acts on $\mathbb{P}(V_\psi^\vee)$ as $\mathrm{PSp}(V_\psi)$ when $\psi^2 = 1$ and as $\mathrm{PSL}(V_\psi)$ when $\psi^2 \neq 1$.

Contents

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1. Introduction

Let

$$\Gamma_g = \pi_1(\Sigma_g), \qquad g \geq 2,$$

where Σ_g is a closed connected oriented surface. A finite character

$$\psi : \Gamma_g \longrightarrow \mathbb{C}^\times$$

factors through $H_1(\Sigma_g;\mathbb{Z})$ and hence through a finite cyclic quotient. Let $\mathbb{C}_\psi$ be the corresponding rank-one local system. If $\psi \neq 1$, then

$$H^0(\Sigma_g;\mathbb{C}_\psi) = 0, \qquad H^2(\Sigma_g;\mathbb{C}_\psi) = 0,$$

the second equality following from Poincaré duality and the vanishing of $H^0(\Sigma_g;\mathbb{C}_{\psi^{-1}})$. Therefore

$$\dim_{\mathbb{C}} H^1(\Sigma_g;\mathbb{C}_\psi) = 2g-2. \tag{1}$$

We write

$$V_\psi := H^1(\Sigma_g;\mathbb{C}_\psi).$$

The first point of this paper is that the full Prym cohomology space gives a rank $2g-1$ affine-unipotent example. More precisely, the identity tensor

$$\mathrm{id}_{V_\psi} \in V_\psi \otimes V_\psi^\vee$$

defines a triangular representation of rank

$$1 + \dim V_\psi = 2g-1$$

with infinite image and finite ordinary mapping-class-group orbit. This is the existence, or upper-bound, part of the paper.

The second point is a restricted optimality statement. We consider only scalar two-block triangular Prym extensions: representations whose semisimplification has exactly two scalar isotypical types, copies of $\mathbb{C}_\psi$ and copies of the trivial character. In this restricted category, the rank $2g-1$ construction cannot be compressed.

The lower-bound argument has two separate parts. First, Looijenga's cyclic Prym theorem is converted into a single-character projective Zariski-closure statement for the ordinary stabilizer H_ψ. This requires keeping track of the cyclic quotient, the lifted Prym group, deck transformations, and the homology–cohomology duality convention. Second, the extension class is converted into a constant-rank linear space of matrices, where Westwick's fixed-rank bound applies.

Main results.

Theorem 1.1 (Rank $2g-1$ affine-Prym examples)**.** *Let $g \geq 2$ and let $\psi : \pi_1(\Sigma_g) \to \mathbb{C}^\times$ be a non-trivial finite character. Set $V_\psi = H^1(\Sigma_g; \mathbb{C}_\psi)$. There exists a representation*

$$\rho_\psi^{\mathrm{univ}} : \pi_1(\Sigma_g) \longrightarrow \mathrm{GL}_{2g-1}(\mathbb{C})$$

of the form

$$\rho_\psi^{\mathrm{univ}}(\gamma) = \begin{pmatrix} \psi(\gamma) I_{V_\psi^\vee} & U(\gamma) \\ 0 & 1 \end{pmatrix},$$

where U is a cocycle representing the identity tensor in $V_\psi \otimes V_\psi^\vee$, such that:

(a) *$\rho_\psi^{\mathrm{univ}}$ has rank $1 + \dim V_\psi = 2g-1$;*
(b) *its image is infinite;*
(c) *its ordinary $\mathrm{Mod}(\Sigma_g)$-orbit is finite.*

The same construction applies to a punctured surface $\Sigma_{g,n}$, under Convention 2.6, with V_ψ replaced by $W_1 H^1(\Sigma_{g,n}; \mathbb{C}_\psi) \cong H^1(\Sigma_g; \mathbb{C}_\psi)$.

Theorem 1.2 (Restricted scalar two-block optimality)**.** *Let $g \geq 2$, let $\psi : \pi_1(\Sigma_g) \to \mathbb{C}^\times$ be a non-trivial finite character, and put $V_\psi = H^1(\Sigma_g; \mathbb{C}_\psi)$. Let*

$$0 \longrightarrow \mathbb{C}_\psi \otimes M \longrightarrow \rho \longrightarrow \mathbb{C} \otimes N \longrightarrow 0$$

be a scalar two-block triangular Prym extension with $M, N \neq 0$. Assume that:

(i) *the ordinary $\mathrm{Mod}(\Sigma_g)$-orbit of the conjugacy class of ρ is finite;*
(ii) *the image of ρ is infinite.*

Then

$$\dim M + \dim N \geq \dim V_\psi + 1 = 2g - 1.$$

Consequently, the rank $2g-1$ affine-Prym construction of Theorem 1.1 is optimal inside the scalar two-block triangular Prym category, and no global optimality outside this category is asserted.

Remark 1.3 (Scope of the optimality statement)*.* The optimality statement is deliberately restricted to scalar two-block triangular Prym extensions. It does not address representations with non-scalar finite-image semisimple blocks, longer non-semisimple filtrations, semisimple finite-orbit representations, point-pushing constructions, quantum or TQFT constructions, or other mechanisms outside this class.

2. The affine-Prym construction

This section proves Theorem 1.1. We begin with the closed case and then record the punctured version under peripheral triviality.

2.1. The universal extension class.

Let

$$V_\psi = H^1(\Sigma_g; \mathbb{C}_\psi), \qquad M := V_\psi^\vee, \qquad N := \mathbb{C}.$$

Then

$$\mathrm{Ext}^1_{\Gamma_g}(\mathbb{C}, \mathbb{C}_\psi \otimes M) \cong H^1(\Sigma_g; \mathbb{C}_\psi) \otimes M = V_\psi \otimes V_\psi^\vee = \mathrm{End}(V_\psi).$$

Let

$$x_{\mathrm{univ}} := \mathrm{id}_{V_\psi} \in V_\psi \otimes V_\psi^\vee.$$

Choose a cocycle

$$U \in Z^1(\Gamma_g; \mathbb{C}_\psi \otimes V_\psi^\vee)$$

representing x_{univ}. Thus

$$U(\gamma\delta) = U(\gamma) + \psi(\gamma) U(\delta).$$

Define

$$\rho_\psi^{\mathrm{univ}}(\gamma) = \begin{pmatrix} \psi(\gamma) I_{V_\psi^\vee} & U(\gamma) \\ 0 & 1 \end{pmatrix}.$$

Here $U(\gamma)$ is viewed as a linear map $\mathbb{C} \to V_\psi^\vee$.

Lemma 2.1 (Representation identity). *The assignment $\gamma \mapsto \rho_\psi^{\mathrm{univ}}(\gamma)$ is a representation of Γ_g.*

Proof. Multiplying the two matrices gives

$$\begin{pmatrix} \psi(\gamma) I & U(\gamma) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \psi(\delta) I & U(\delta) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \psi(\gamma\delta) I & U(\gamma) + \psi(\gamma) U(\delta) \\ 0 & 1 \end{pmatrix}.$$

The upper-right entry is $U(\gamma\delta)$ by the cocycle identity. □

2.2. Infinite image.

Lemma 2.2 (Non-vanishing on the kernel). *Let $K = \ker \psi$. For any cocycle U representing the non-zero class x_{univ}, the restriction $U|_K$ is not identically zero.*

Proof. Suppose $U|_K = 0$. Since K acts trivially on the coefficient module $\mathbb{C}_\psi \otimes V_\psi^\vee$, the cocycle U descends to a cocycle of the finite group $\Gamma_g/K = \mathrm{im}(\psi)$. Over characteristic zero, the cohomology of a finite group with coefficients in a finite-dimensional complex representation vanishes in positive degree. Hence the descended cocycle is a coboundary. Pulling back, U is a coboundary on Γ_g, contradicting the fact that $x_{\mathrm{univ}} \neq 0$ in $H^1(\Sigma_g; \mathbb{C}_\psi) \otimes V_\psi^\vee$. □

Corollary 2.3 (Infinite image)**.** *The image of* $\rho_\psi^{\mathrm{univ}}$ *is infinite.*

Proof. By Lemma 2.2, choose $\gamma \in K$ such that $U(\gamma) \neq 0$. Then $\psi(\gamma) = 1$ and
$$\rho_\psi^{\mathrm{univ}}(\gamma) = \begin{pmatrix} I & U(\gamma) \\ 0 & 1 \end{pmatrix}$$
is a non-trivial unipotent element. Its powers have upper-right entry $nU(\gamma)$, so they are pairwise distinct. □

2.3. **Finite ordinary mapping-class-group orbit.** We use ordinary conjugacy throughout: no projectivization, twisting equivalence, or GIT quotient is imposed.

Lemma 2.4 (Stabilizer fixes the universal class up to ordinary conjugacy)**.** *Let*
$$H_\psi = \mathrm{Stab}_{\mathrm{Mod}(\Sigma_g)}(\psi).$$
Then the ordinary conjugacy class of $\rho_\psi^{\mathrm{univ}}$ *is fixed by* H_ψ*.*

Proof. Let
$$A_h : V_\psi \longrightarrow V_\psi$$
be the linear automorphism induced by h. Since $h \in H_\psi$, the transformed extension class lies again in $V_\psi \otimes V_\psi^\vee$. Under the identification
$$V_\psi \otimes V_\psi^\vee \cong \mathrm{End}(V_\psi),$$
the universal class is the identity. Equivalently, the contraction map
$$T_{x_{\mathrm{univ}}} : V_\psi^\vee \longrightarrow M = V_\psi^\vee$$
is the identity map. After applying h on the Prym factor, the contraction map becomes $A_h^\vee$. Now change the basis of the multiplicity space $M = V_\psi^\vee$ by $(A_h^\vee)^{-1}$. On the representation this is the block-diagonal change of basis
$$P_h = \begin{pmatrix} (A_h^\vee)^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$
It sends the transformed contraction map back to $(A_h^\vee)^{-1}A_h^\vee = \mathrm{id}_{V_\psi^\vee}$. If the chosen transformed cocycle representative differs from the original representative U by a coboundary, the remaining difference is removed by the usual upper-triangular change of splitting. Hence the transformed representation is ordinarily conjugate to $\rho_\psi^{\mathrm{univ}}$. □

Lemma 2.5 (Finite character orbit)**.** *The* $\mathrm{Mod}(\Sigma_g)$*-orbit of a finite character* ψ *is finite.*

Proof. If the order of ψ divides q, then ψ factors through the finite group $H_1(\Sigma_g; \mathbb{Z}/q\mathbb{Z})$. There are only finitely many characters of this finite group. □

Proof of Theorem 1.1 in the closed case. The rank is $1 + \dim V_\psi = 2g - 1$. Infinite image follows from Corollary 2.3. By Lemma 2.5, the character orbit is finite. For each character in that finite orbit, its stabilizer fixes the corresponding universal ordinary conjugacy class by Lemma 2.4. Therefore the full ordinary mapping-class-group orbit of $\rho_\psi^{\mathrm{univ}}$ is finite. □

2.4. **Punctured surfaces under peripheral triviality.** Let $\Sigma_{g,n}$ be a punctured surface, and let $\bar{\Sigma}_{g,n}$ denote the closed surface obtained by filling the punctures. Thus $\bar{\Sigma}_{g,n} \cong \Sigma_g$.

Convention 2.6 (Peripheral triviality)**.** *For a punctured surface $\Sigma_{g,n}$, assume that ψ is trivial on peripheral loops and that the triangular representation has trivial peripheral monodromy. In matrix form, this means that the cocycle U vanishes on each puncture loop.*

Under Convention 2.6, the punctured statement is only the filled-surface, weight-one consequence of the closed theorem. The relevant cohomology is

$$W_1 H^1(\Sigma_{g,n}; \mathbb{C}_\psi) \cong H^1(\bar{\Sigma}_{g,n}; \mathbb{C}_\psi) \cong H^1(\Sigma_g; \mathbb{C}_\psi).$$

This identification is equivariant for the natural map from the mapping class group of the punctured surface to the mapping class group of the filled surface: filling the punctures forgets peripheral loops, and any permutation of punctures acts trivially on the filled Prym block. The fundamental group of the filled surface is the quotient of $\pi_1(\Sigma_{g,n})$ by the normal subgroup generated by the peripheral loops. Under Convention 2.6, each peripheral loop c satisfies

$$\psi(c) = 1, \qquad U(c) = 0,$$

and therefore $\rho(c) = I$. Hence the triangular representation kills the normal subgroup generated by the peripheral loops and factors through $\pi_1(\bar{\Sigma}_{g,n})$. Mapping classes in the filling kernel act on this quotient by inner automorphisms, or trivially after forgetting the peripheral data; ordinary conjugacy classes are insensitive to such inner changes. Thus the punctured ordinary orbit is the pullback of the corresponding filled-surface ordinary orbit. Applying the closed construction to this weight-one block gives the same rank

$$1 + \dim W_1 H^1(\Sigma_{g,n}; \mathbb{C}_\psi) = 2g - 1.$$

Since the factored representation is exactly the filled-surface construction on the weight-one Prym block, it has the same finite-orbit and infinite-image properties as in the closed case. Peripheral non-trivial characters, non-zero peripheral cocycle classes, and the extra boundary part of punctured-surface cohomology are not considered here.

3. Looijenga's cyclic Prym theorem

The Prym monodromy needed for the lower bound is Looijenga's cyclic image theorem, applied to the cyclic quotient attached to the character ψ.

3.1. **The cyclic quotient attached to a character.** Let $\psi : \pi_1(\Sigma_g) \to \mathbb{C}^\times$ be finite and non-trivial. It factors through $H_1(\Sigma_g;\mathbb{Z})$. Let

$$q : H_1(\Sigma_g;\mathbb{Z}) \twoheadrightarrow C$$

be the finite cyclic quotient determined by ψ, so that pullback identifies

$$q^*(\widehat{C}) = \langle\psi\rangle \subset \operatorname{Hom}(H_1(\Sigma_g;\mathbb{Z}),\mathbb{C}^\times), \qquad \widehat{C} := \operatorname{Hom}(C,\mathbb{C}^\times).$$

Equivalently, C is the cyclic deck group of the connected cyclic cover that realizes the ψ-Prym eigenspace.

Let

$$H_\psi := \operatorname{Stab}_{\operatorname{Mod}(\Sigma_g)}(\psi).$$

Looijenga's subgroup associated with C, denoted here by S_C, is the subgroup of mapping classes inducing the identity on the quotient C. This is equivalently the subgroup fixing $\widehat{C}$ pointwise under the dual pullback action. Since $q^*(\widehat{C}) = \langle\psi\rangle$ is generated by ψ, fixing ψ is equivalent to fixing every power of ψ, hence to fixing $q^*(\widehat{C})$ pointwise. Thus, in this single-character setup,

$$S_C = H_\psi.$$

The point here is the identity action on the quotient C, not merely preservation of the kernel of q or preservation of the corresponding cover.

3.2. **Looijenga's cyclic image theorem.** If $S_C^\#$ denotes the group of lifts of mapping classes inducing the identity on the cyclic quotient C, then there is a central extension

$$1 \longrightarrow C \longrightarrow S_C^\# \longrightarrow S_C \longrightarrow 1,$$

where the kernel is the deck group. Here $S_C^\#$ is our abbreviation for Looijenga's lifted group $\mathcal{S}^\#_{;C}$.

Theorem 3.1 (Looijenga's cyclic Prym image theorem)**.** *Let C be a nontrivial finite cyclic quotient of $H_1(\Sigma_g;\mathbb{Z})$, let $\widetilde{\Sigma}_C \to \Sigma_g$ be the associated connected cyclic cover, and let H_C be Looijenga's cyclic Prym module over the cyclotomic integer ring R_C. Then H_C is a free R_C-module of rank $2(g-1)$ with its standard skew-Hermitian intersection form. Moreover, the image of the lifted mapping class group $S_C^\#$ on H_C is the arithmetic group denoted by $U^\#(H_C)$.*

All notation in Theorem 3.1 follows Looijenga, Section 2. The freeness and form statement are Looijenga's Proposition 2.2 [3, §2]; the finite-squares description of $U^\#(H_C)$ is the paragraph immediately following that proposition; and the image statement is the cyclic case of Looijenga's Theorem 2.4 [3, §2].

Remark 3.2 (Cyclic case). Only the cyclic factor C attached to $\langle\psi\rangle$ is used in what follows.

3.3. **Deck transformations disappear projectively.** Let

$$E_\psi := H_C \otimes_{R_C,\psi} \mathbb{C}$$

be the complex ψ-eigenspace after scalar extension through the embedding determined by ψ. A deck transformation $c \in C$ acts on E_ψ by the scalar $\psi(c)$, or by $\psi(c)^{-1}$, depending on the left/right module convention. In either convention the action is scalar. Hence the deck kernel acts trivially on $\mathbb{P}(E_\psi)$.

Thus the projective image of the lifted group $S_C^{\#}$ on $\mathbb{P}(E_\psi)$ and the projective image of the ordinary stabilizer $S_C = H_\psi$ have the same identity component of their projective Zariski closures.

3.4. **Homology versus cohomology.** With the convention used here, the ψ-eigenspace of the homology of the cyclic cover identifies with twisted homology with inverse character:

$$E_\psi \cong H_1(\Sigma_g; \mathbb{C}_{\psi^{-1}}).$$

By Poincaré duality and the universal coefficient theorem over $\mathbb{C}$,

$$E_\psi^\vee \cong H^1(\Sigma_g; \mathbb{C}_\psi) = V_\psi.$$

Thus $E_\psi \cong V_\psi^\vee$ after fixing the standard dual convention. If the opposite left/right convention is used, replace ψ by ψ^{-1}; this exchanges the two conjugate eigenspaces and does not affect the final projective closure conclusion, since dualizing preserves PSp / PSL and the dichotomy $\psi^2 = 1$ versus $\psi^2 \neq 1$.

3.5. **The single-character projective consequence.** The next lemma records the arithmetic passage from Looijenga's K_C/K_C^0-unitary group to one complex character block. It is the only place where Borel density is used: finite determinant and root-of-unity conditions may change component groups, but not the connected projective Zariski closure.

Lemma 3.3 (Arithmetic density and base change)**.** *Write $K = K_C$ and $K^+ = K_C^0$. Let $\mathbf{SU}(H_C)$ be the K^+-algebraic special unitary group of Looijenga's skew-Hermitian R_C-module H_C, and set*

$$\mathbf{G}_{\mathbb{Q}} = \mathrm{Res}_{K^+/\mathbb{Q}}\,\mathbf{SU}(H_C).$$

Let Γ be the special-unitary arithmetic subgroup determined by the R_C-lattice H_C. After passing from $\mathbf{G}_{\mathbb{Q}}(\mathbb{R})$ to its non-compact semisimple factors, the image of Γ is Zariski dense in those factors. Consequently:

(a) *replacing Γ by a finite-index subgroup does not change the identity component of its complex projective Zariski closure;*
(b) *adjoining the finite cosets in $U^{\#}(H_C)$ coming from determinants and roots of unity does not change that identity component;*

(c) *on a fixed complex character block E_ψ, the identity component of the projective Zariski closure is obtained from the connected complex group coming from*

$$\mathbf{SU}(H_C) \otimes_{K^+,\psi|_{K^+}} \mathbb{C}$$

and then taking its induced projective action on $\mathbb{P}(E_\psi)$.

Proof. The group $\mathbf{G}_\mathbb{Q}$ is a connected semisimple $\mathbb{Q}$-group obtained by restriction of scalars from K^+ to $\mathbb{Q}$. Borel density, applied after discarding compact factors, gives Zariski density of the arithmetic subgroup in the noncompact semisimple real factors; see Borel [1] and the standard treatment in Platonov–Rapinchuk [2]. Passing to a finite-index subgroup only replaces the arithmetic subgroup by finitely many cosets inside its original group, so the identity component of the complex Zariski closure is unchanged. Likewise, Looijenga's $U^\#(H_C)$ differs from the special-unitary arithmetic subgroup by the finite $C^{(2)}$ and determinant/root-of-unity cosets recorded after his Proposition 2.2; these finite cosets can add at most components, not alter the identity component. Finally, base change through the embedding of K^+ determined by $\psi|_{K^+}$ selects the complex factor acting on the block E_ψ, and projectivizing passes to the asserted projective identity component. □

Lemma 3.4 (Classical group on one character block)**.** *Assume the image statement of Theorem 3.1. Let*

$$E_\psi = H_C \otimes_{R_C,\psi} \mathbb{C}$$

be the complex eigenspace attached to a non-trivial character $\psi \in \widehat{C}$. Then the identity component of the projective Zariski closure of $U^\#(H_C)$ on $\mathbb{P}(E_\psi)$ is

$$\begin{cases} \mathrm{PSp}(E_\psi), & \psi^2 = 1, \\ \mathrm{PSL}(E_\psi), & \psi^2 \neq 1. \end{cases}$$

Proof. Write $K = K_C$ and $K^+ = K_C^0$. Looijenga's form on H_C is a nondegenerate skew-Hermitian form over R_C, with respect to the involution of the cyclotomic field. Let $\mathbf{SU}(H_C)$ denote the corresponding K^+-algebraic special unitary group. By Lemma 3.3, for the purpose of computing the identity component of the complex projective Zariski closure, it is enough to base-change $\mathbf{SU}(H_C)$ to $\mathbb{C}$, take the connected complex group on the character block in question, and then projectivize. The finite $C^{(2)}$, determinant, and root-of-unity conditions in $U^\#(H_C)$ affect only components.

First suppose $\psi^2 = 1$. Since $\psi \neq 1$, the character ψ has order two and the block E_ψ is self-dual. On this block the cyclotomic involution is trivial after scalar extension, so the skew-Hermitian relation

$$\langle y, x\rangle = -\overline{\langle x, y\rangle}$$

becomes the complex bilinear skew-symmetry relation

$$\langle y, x\rangle = -\langle x, y\rangle.$$

In characteristic zero this is the same as an alternating form. Looijenga's intersection form is non-degenerate, so its restriction/scalar extension on the self-dual block is a non-degenerate alternating complex bilinear form. Therefore the connected complex group preserving the form on E_ψ is $\mathrm{Sp}(E_\psi)$. It is not an orthogonal group: an orthogonal group would arise from a non-degenerate symmetric bilinear form, whereas here the form is skew-symmetric/alternating. Projectivizing gives $\mathrm{PSp}(E_\psi)$.

Now suppose $\psi^2 \neq 1$. Then ψ and ψ^{-1} are distinct conjugate characters. After base change through the real embedding of K^+ determined by $\psi|_{K^+}$, one has

$$K \otimes_{K^+} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C},$$

with the two factors corresponding to ψ and ψ^{-1}. Hence

$$H_C \otimes_{K^+} \mathbb{C} = E_\psi \oplus E_{\psi^{-1}}.$$

The form is invariant under the deck action, so two eigenspaces can pair nontrivially only when their characters multiply to the trivial character. Since $\psi^2 \neq 1$, the restriction of the form to E_ψ is zero, and the same holds for $E_{\psi^{-1}}$; each summand is isotropic. Non-degeneracy of the total form then gives a perfect pairing

$$E_\psi \times E_{\psi^{-1}} \longrightarrow \mathbb{C}.$$

Consequently the complexified unitary group preserving this pairing acts on the two summands by

$$A \oplus (A^{-1})^\vee.$$

After imposing the connected special/derived condition relevant to the projective identity component, we may take $A \in \mathrm{SL}(E_\psi)$. Thus the connected action on the single block E_ψ is the standard $\mathrm{SL}(E_\psi)$-action, and the induced connected projective group on $\mathbb{P}(E_\psi)$ is $\mathrm{PSL}(E_\psi)$. Equivalently, allowing $A \in \mathrm{GL}(E_\psi)$ gives the same projective image, since over $\mathbb{C}$ the image of $\mathrm{GL}(E_\psi)$ in $\mathrm{PGL}(E_\psi)$ is the same connected adjoint type-A group as $\mathrm{PSL}(E_\psi)$. As above, determinant, norm-one, and root-of-unity restrictions in $U^\#(H_C)$ only affect finite cosets/components and do not cut down this identity component. $\square$

Proposition 3.5 (Single-character projective closure)**.** *Let $\psi : \pi_1(\Sigma_g) \to \mathbb{C}^\times$ be a non-trivial finite character, and set*

$$V_\psi = H^1(\Sigma_g; \mathbb{C}_\psi), \qquad H_\psi = \mathrm{Stab}_{\mathrm{Mod}(\Sigma_g)}(\psi).$$

Let

$$\rho_\psi^{\mathrm{proj}} : H_\psi \longrightarrow \mathrm{PGL}(V_\psi^\vee)$$

be the induced projective representation. Then

$$\left(\overline{\rho_\psi^{\mathrm{proj}}(H_\psi)}^{\mathrm{Zar}}\right)^0 = \begin{cases} \mathrm{PSp}(V_\psi), & \psi^2 = 1, \\ \mathrm{PSL}(V_\psi), & \psi^2 \neq 1, \end{cases}$$

where the groups on the right are viewed through the natural dual projective action on $\mathbb{P}(V_\psi^\vee)$. If $H_0 \le H_\psi$ is a finite-index subgroup, then

$$\left(\overline{\rho_\psi^{\mathrm{proj}}(H_0)}^{\mathrm{Zar}}\right)^0 = \left(\overline{\rho_\psi^{\mathrm{proj}}(H_\psi)}^{\mathrm{Zar}}\right)^0.$$

Proof. We spell out the translation from Looijenga's notation to the projective action used here.

(L1) **The quotient and the stabilizer.** Choose the cyclic quotient $q : H_1(\Sigma_g;\mathbb{Z}) \twoheadrightarrow C$ determined by ψ, so that

$$q^*(\widehat{C}) = \langle \psi \rangle \subset \mathrm{Hom}(H_1(\Sigma_g;\mathbb{Z}), \mathbb{C}^\times).$$

Looijenga's subgroup S_C is the subgroup inducing the identity on the quotient C. By duality this is the subgroup fixing $\widehat{C}$ pointwise; after pullback it fixes every power of ψ. Since ψ generates $q^*(\widehat{C})$, this is exactly $H_\psi = \mathrm{Stab}(\psi)$. This is stronger than preserving only the kernel of q.

(L2) **The lifted group and deck kernel.** Looijenga's action is first defined for the lifted group

$$1 \longrightarrow C \longrightarrow S_C^\# \longrightarrow S_C \longrightarrow 1.$$

The kernel is the deck group. On $E_\psi = H_C \otimes_{R_C,\psi} \mathbb{C}$, a deck transformation $c \in C$ acts by $\psi(c)$ or by $\psi(c)^{-1}$, depending on the left/right module convention. In either convention the action is scalar, so it is trivial in $\mathrm{PGL}(E_\psi)$. Therefore passing from $S_C^\#$ to $S_C = H_\psi$ does not change the connected projective Zariski closure.

(L3) **The relevant complex block.** With our convention,

$$E_\psi \cong H_1(\Sigma_g;\mathbb{C}_{\psi^{-1}}), \qquad E_\psi^\vee \cong H^1(\Sigma_g;\mathbb{C}_\psi) = V_\psi.$$

Thus E_ψ is the dual vector space on which the projective action is written as $\mathbb{P}(V_\psi^\vee)$. Replacing ψ by ψ^{-1} under the opposite convention only dualizes or conjugates the same projective statement and leaves the dichotomy $\psi^2 = 1$ versus $\psi^2 \ne 1$ unchanged.

(L4) **The image and the connected complex group.** By Theorem 3.1, the image of $S_C^\#$ on H_C is $U^\#(H_C)$. Lemma 3.4 computes the identity component of the complex projective Zariski closure of this arithmetic group on the single block E_ψ: it is $\mathrm{PSp}(E_\psi)$ in the self-dual order-two case and $\mathrm{PSL}(E_\psi)$ in the non-self-dual case. The arithmetic Zariski-density/base-change step used in that computation is isolated in Lemma 3.3.

Combining (L1)–(L4) gives the displayed formula, after identifying $E_\psi \cong V_\psi^\vee$ and using the natural dual projective action.

For the finite-index assertion, let $H_0 \le H_\psi$ be a finite-index subgroup. Then $\rho_\psi^{\mathrm{proj}}(H_\psi)$ is a finite union of cosets of $\rho_\psi^{\mathrm{proj}}(H_0)$. The Zariski closure of a finite union of cosets is a finite union of translates of $\overline{\rho_\psi^{\mathrm{proj}}(H_0)}^{\mathrm{Zar}}$.

Finite unions of translates may add components, but they do not change the identity component. Hence the two connected projective Zariski closures are equal. □

Corollary 3.6 (Finite-orbit projective subvarieties are trivial). *Let $Z \subset \mathbb{P}(V_\psi^\vee)$ be a closed subvariety. If the H_ψ-orbit of Z is finite, then*

$$Z = \varnothing \qquad \text{or} \qquad Z = \mathbb{P}(V_\psi^\vee).$$

The same conclusion holds with H_ψ replaced by any finite-index subgroup.

Proof. If the orbit of Z is finite, some finite-index subgroup $H_0 \le H_\psi$ stabilizes Z. The stabilizer of a closed projective subvariety is Zariski closed in the relevant projective linear group. By Proposition 3.5, the connected projective Zariski closure of H_0 is either $\mathrm{PSp}(V_\psi)$ or $\mathrm{PSL}(V_\psi)$. Both groups act transitively on $\mathbb{P}(V_\psi^\vee)$. For PSL this is standard; for PSp, every non-zero vector in a symplectic vector space is isotropic and can be included in a symplectic basis, so any two lines can be carried to one another by a symplectic automorphism. Hence Z is either empty or the whole projective space. □

4. Extension classes and ordinary conjugacy

Fix non-zero finite-dimensional complex vector spaces M and N, and put

$$m = \dim M, \qquad n = \dim N, \qquad U = \mathrm{Hom}(N, M).$$

A scalar two-block triangular Prym extension is an extension

$$0 \longrightarrow \mathbb{C}_\psi \otimes M \longrightarrow \rho \longrightarrow \mathbb{C} \otimes N \longrightarrow 0.$$

After choosing a vector-space splitting, it has the form

$$\rho(\gamma) = \begin{pmatrix} \psi(\gamma) I_M & u(\gamma) \\ 0 & I_N \end{pmatrix},$$

where u is a cocycle with values in $\mathbb{C}_\psi \otimes \mathrm{Hom}(N, M)$. The extension class lies in

$$\mathrm{Ext}^1_{\Gamma_g}(\mathbb{C} \otimes N, \mathbb{C}_\psi \otimes M) \cong H^1(\Sigma_g; \mathbb{C}_\psi) \otimes \mathrm{Hom}(N, M) = V_\psi \otimes U.$$

Write this class as

$$x \in V_\psi \otimes U.$$

Changing bases in M and N gives the group

$$K = \mathrm{GL}(M) \times \mathrm{GL}(N),$$

acting on $U = \mathrm{Hom}(N, M)$ by

$$(A, B) \cdot T = ATB^{-1}.$$

It acts on $V_\psi \otimes U$ through the second tensor factor. The notation $(V_\psi \otimes U)/K$ denotes the set of ordinary K-orbits, not a GIT quotient.

Lemma 4.1 (Infinite image and non-zero extension class)**.** *For a scalar two-block triangular Prym extension, the image of ρ is finite if and only if its extension class x is zero. In particular, infinite image implies $x \neq 0$.*

Proof. If $x = 0$, the extension splits as $(\mathbb{C}_\psi \otimes M) \oplus (\mathbb{C} \otimes N)$. Both diagonal characters have finite image, so the image of ρ is finite.

Conversely, if the image of ρ is finite, then ρ factors through a finite group. Over $\mathbb{C}$, every representation of a finite group is semisimple by Maschke's theorem. Hence the extension splits, and its extension class is zero. □

Lemma 4.2 (Ordinary conjugacy and K-orbits)**.** *Fix the semisimplification*

$$(\mathbb{C}_\psi \otimes M) \oplus (\mathbb{C} \otimes N).$$

Two scalar two-block extensions with classes $x, x' \in V_\psi \otimes \operatorname{Hom}(N, M)$ are isomorphic as Γ_g-local systems if and only if x' lies in the $K = \mathrm{GL}(M) \times \mathrm{GL}(N)$-orbit of x.

Proof. Let $A = \mathbb{C}_\psi \otimes M$ and $B = \mathbb{C} \otimes N$. Since $\psi \neq 1$, the two rank-one local systems $\mathbb{C}_\psi$ and $\mathbb{C}$ are non-isomorphic. Equivalently,

$$\operatorname{Hom}_{\Gamma_g}(B, A) = H^0(\Gamma_g; \mathbb{C}_\psi) \otimes \operatorname{Hom}(N, M) = 0,$$

and similarly

$$\operatorname{Hom}_{\Gamma_g}(A, B) = H^0(\Gamma_g; \mathbb{C}_{\psi^{-1}}) \otimes \operatorname{Hom}(M, N) = 0.$$

Thus an isomorphism of local systems cannot mix the two scalar isotypical constituents in the semisimplification. It preserves the $\mathbb{C}_\psi$-isotypical summand and the trivial isotypical summand, and therefore induces automorphisms of the multiplicity spaces M and N. These automorphisms form

$$\operatorname{Aut}_{\Gamma_g}(A) \times \operatorname{Aut}_{\Gamma_g}(B) = \mathrm{GL}(M) \times \mathrm{GL}(N) = K.$$

After a vector-space splitting, an extension is represented by a cocycle $u \in Z^1(\Gamma_g; \mathbb{C}_\psi \otimes \operatorname{Hom}(N, M))$. Changing the splitting replaces u by $u + d\phi$, hence does not change the cohomology class in $\operatorname{Ext}^1_{\Gamma_g}(B, A)$. Changing the bases of M and N by $(A_0, B_0) \in K$ sends the cohomology class to $(A_0, B_0) \cdot x$, where the action on $U = \operatorname{Hom}(N, M)$ is $T \mapsto A_0 T B_0^{-1}$. Conversely, any element of K is realized by such a change of multiplicity bases and gives an ordinary isomorphism of the resulting extensions. Hence ordinary representation isomorphism classes with the fixed semisimplification are exactly ordinary K-orbits in $V_\psi \otimes U$. No GIT quotient, projectivization, or twisting equivalence is being used. □

Lemma 4.3 (Finite ordinary orbit gives finite K-orbit data)**.** *If the ordinary $\operatorname{Mod}(\Sigma_g)$-orbit of the conjugacy class of ρ is finite, then the K-orbit*

$$[x] \in (V_\psi \otimes U)/K$$

has finite orbit under H_ψ.

Proof. Restrict the finite $\mathrm{Mod}(\Sigma_g)$-orbit to H_ψ. For every $h \in H_\psi$, the pulled-back representation has semisimplification

$$(\mathbb{C}_\psi \otimes M) \oplus (\mathbb{C} \otimes N),$$

because h fixes ψ and fixes the trivial character. Thus all representations in the restricted orbit lie in the same fixed two-isotypical category. By Lemma 4.2, each ordinary conjugacy class in this category determines a single K-orbit of extension classes, and ordinary isomorphism is exactly K-orbit equivalence. Since only finitely many ordinary conjugacy classes occur in the original mapping-class orbit, only finitely many K-orbits occur under H_ψ. □

5. Full Prym support

Definition 5.1 (Prym support). For $x \in V_\psi \otimes U$, define its support in the Prym factor by

$$S(x) := \mathrm{span}\{(\mathrm{id}_{V_\psi} \otimes \lambda)(x) : \lambda \in U^\vee\} \subset V_\psi.$$

Equivalently, $S(x)$ is the smallest linear subspace $S \subset V_\psi$ such that $x \in S \otimes U$.

Lemma 5.2 (Equivariance of support). *For $k \in K$ and $h \in H_\psi$,*

$$S(kx) = S(x), \qquad S(hx) = hS(x).$$

Proof. The group K acts invertibly on the U-factor and trivially on the V_ψ-factor, so contraction by all elements of $U^\vee$ spans the same subspace. The mapping class group acts on the extension class through the Prym factor, so contraction in U commutes with the induced action on V_ψ. □

Lemma 5.3 (No finite-index invariant proper subspace). *No finite-index subgroup of H_ψ fixes a non-zero proper linear subspace of V_ψ.*

Proof. Suppose $0 \neq W \subsetneq V_\psi$ is fixed by a finite-index subgroup $H_0 \leq H_\psi$. Its annihilator $W^\perp \subset V_\psi^\vee$ is non-zero and proper. Therefore $\mathbb{P}(W^\perp)$ is a non-empty proper closed subvariety of $\mathbb{P}(V_\psi^\vee)$. Since H_0 is finite-index in H_ψ, the H_ψ-orbit of $\mathbb{P}(W^\perp)$ is finite. This contradicts Corollary 3.6. □

Proposition 5.4 (Full Prym support). *Let $x \in V_\psi \otimes U$ be non-zero. If the K-orbit $[x]$ has finite H_ψ-orbit, then*

$$S(x) = V_\psi.$$

Proof. Since $[x]$ has finite H_ψ-orbit, there is a finite-index subgroup $H_0 \leq H_\psi$ stabilizing $[x]$. Thus for every $h \in H_0$ there exists $k_h \in K$ such that $hx = k_h x$. By Lemma 5.2,

$$hS(x) = S(hx) = S(k_h x) = S(x).$$

Thus $S(x)$ is fixed by H_0. Since $x \neq 0$, $S(x) \neq 0$: if all contractions by $U^\vee$ vanished, then the standard finite-dimensional identification $V_\psi \otimes U \cong$

$\mathrm{Hom}(U^\vee, V_\psi)$ would give $x = 0$. By Lemma 5.3, $S(x)$ cannot be a proper non-zero subspace. Hence $S(x) = V_\psi$. □

Corollary 5.5 (Injectivity of the contraction map)**.** *Under the hypotheses of Proposition 5.4, the contraction map*

$$T_x : V_\psi^\vee \longrightarrow U = \mathrm{Hom}(N, M)$$

associated with x is injective.

Proof. The dual map $T_x^\vee : U^\vee \to V_\psi$ has image exactly $S(x)$. Thus $S(x) = V_\psi$ is equivalent to surjectivity of $T_x^\vee$, which is equivalent to injectivity of T_x. □

6. Rank strata and constant rank

Assume T_x is injective and set

$$L := T_x(V_\psi^\vee) \subset U = \mathrm{Hom}(N, M).$$

Then

$$\dim L = \dim V_\psi = 2g - 2.$$

We use throughout the contraction convention recorded in Appendix A: if h acts on V_ψ and contragrediently on $V_\psi^\vee$, then

$$T_{hx}(\ell) = T_x(h^{-1}\ell).$$

Thus the formulas below are statements about ordinary projective orbits; choosing the opposite left/right convention replaces h by h^{-1} and leaves the finite-orbit argument unchanged.

Definition 6.1 (Pulled-back determinantal strata)*.* For $0 \le r \le \min(m, n)$, let

$$D_r = \{A \in \mathrm{Hom}(N, M) : \mathrm{rank}\, A \le r\}$$

be the usual determinantal variety. Define

$$Z_r(x) = \{[\ell] \in \mathbb{P}(V_\psi^\vee) : \mathrm{rank}\, T_x(\ell) \le r\}.$$

This is a closed subvariety of $\mathbb{P}(V_\psi^\vee)$ because D_r is cut out by the $(r+1) \times (r+1)$ minors.

Lemma 6.2 (Rank strata are K-orbit invariants)**.** *For every $k \in K$ and every r,*

$$Z_r(kx) = Z_r(x).$$

Proof. Write $k = (A, B) \in \mathrm{GL}(M) \times \mathrm{GL}(N)$. The map attached to kx is

$$T_{kx}(\ell) = AT_x(\ell)B^{-1}.$$

Left and right multiplication by invertible matrices preserve rank. □

Lemma 6.3 (Finite orbit of rank strata)**.** *If the K-orbit $[x]$ has finite H_ψ-orbit, then each $Z_r(x)$ has finite H_ψ-orbit as a closed subvariety of $\mathbb{P}(V_\psi^\vee)$.*

Proof. Using the convention fixed in Appendix A, if h acts on V_ψ on the left, then

$$T_{hx}(\ell) = T_x(h^{-1}\ell)$$

for the contragredient action on $V_\psi^\vee$. Hence

$$Z_r(hx) = hZ_r(x).$$

By Lemma 6.2, replacing hx by a K-equivalent tensor does not change the rank stratum. Since the H_ψ-orbit of $[x]$ is finite, only finitely many closed subvarieties Z_r can occur. □

Proposition 6.4 (Finite-orbit rank strata force constant rank)**.** *Let $x \in V_\psi \otimes U$ have finite H_ψ-orbit in $(V_\psi \otimes U)/K$, and suppose*

$$T_x : V_\psi^\vee \longrightarrow U$$

is injective. Then every non-zero element of

$$L = T_x(V_\psi^\vee)$$

has the same rank.

Proof. By Lemma 6.3, each $Z_r(x)$ has finite H_ψ-orbit. By Corollary 3.6, each $Z_r(x)$ is either empty or all of $\mathbb{P}(V_\psi^\vee)$. Since T_x is injective, $T_x(\ell) \neq 0$ whenever $0 \neq \ell \in V_\psi^\vee$.

If the non-zero elements of L do not all have the same rank, let

$$r_0 = \min\{\operatorname{rank} A : 0 \neq A \in L\}.$$

Then $Z_{r_0}(x)$ is non-empty, but it is not all of $\mathbb{P}(V_\psi^\vee)$ because some non-zero element of L has rank larger than r_0. This contradicts Corollary 3.6. □

7. Westwick bound and proof of optimality

Theorem 7.1 (Westwick fixed-rank bound)**.** *Let $2 \leq r \leq m \leq n$. If*

$$L \subset \operatorname{Hom}(\mathbb{C}^n, \mathbb{C}^m)$$

is a complex linear subspace such that every non-zero element of L has rank exactly r, then

$$\dim L \leq m + n - 2r + 1.$$

This is Westwick's theorem on p. 171, part (1), in the notation $l(r, m, n)$ of [4, p. 171]. The proof of this inequality is completed on p. 172. The hypothesis $m \leq n$ is harmless: if a constant-rank space lies in $\operatorname{Hom}(\mathbb{C}^n, \mathbb{C}^m)$ with $m > n$, the transpose map identifies it linearly and rank-preservingly with a subspace of $\operatorname{Hom}(\mathbb{C}^m, \mathbb{C}^n)$, whose target dimension is then the smaller one.

Lemma 7.2 (Weak constant-rank bound)**.** *Let*

$$L \subset \operatorname{Hom}(\mathbb{C}^n, \mathbb{C}^m)$$

be a complex linear subspace such that every non-zero element of L has the same rank $r \geq 1$. Then

$$\dim L \leq m + n - 1.$$

Proof. If $r \geq 2$, transpose all matrices if necessary so that the smaller of m, n is the target dimension. Theorem 7.1 gives

$$\dim L \leq m + n - 2r + 1 \leq m + n - 3 \leq m + n - 1.$$

It remains to treat $r = 1$. Choose a non-zero element $A = a \otimes \alpha \in L$, with $a \in \mathbb{C}^m$ and $\alpha \in (\mathbb{C}^n)^\vee$. Every non-zero element of L has rank one, and zero has rank at most one. Hence for any $B = b \otimes \beta \in L$, every linear combination $A + tB$ has rank at most one. For rank-one operators this is equivalent to

$$(a \wedge b) \otimes (\alpha \wedge \beta) = 0.$$

Thus for every $B = b \otimes \beta \in L$, either b is proportional to a, or β is proportional to α.

One of these alternatives holds uniformly for all $B \in L$. Indeed, if there were $B = b \otimes \beta$ with b not proportional to a and $C = c \otimes \gamma$ with γ not proportional to α, then the previous paragraph forces β proportional to α and c proportional to a. The sum $B + C$ would then have two independent image directions and two independent covectors, hence rank two, unless one of the two summands is zero; after discarding zero summands this contradicts the fixed rank-one hypothesis. Therefore either all elements of L have image contained in the fixed line $\mathbb{C}a$, or all elements of L factor through the fixed one-dimensional quotient defined by α. In the first case $L \subset \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}a)$, so $\dim L \leq n$. In the second case $L \subset \mathrm{Hom}(\mathbb{C}^n / \ker \alpha, \mathbb{C}^m)$, so $\dim L \leq m$. Hence

$$\dim L \leq \max(m, n) \leq m + n - 1,$$

since $m, n \geq 1$ for a non-zero rank-one space. □

Proof of Theorem 1.2. Let $x \in V_\psi \otimes U$ be the extension class of ρ. By Lemma 4.1, the infinite-image assumption gives $x \neq 0$. By Lemma 4.3, the ordinary finite mapping-class-group orbit gives a finite H_ψ-orbit of the K-orbit $[x]$. Therefore Proposition 5.4 gives $S(x) = V_\psi$, and Corollary 5.5 gives an injection

$$T_x : V_\psi^\vee \hookrightarrow \mathrm{Hom}(N, M).$$

Set

$$L = T_x(V_\psi^\vee).$$

Then

$$\dim L = \dim V_\psi = 2g - 2.$$

By Proposition 6.4, every non-zero element of L has the same rank. By Lemma 7.2,

$$2g - 2 = \dim L \leq \dim M + \dim N - 1.$$

Hence

$$\dim M + \dim N \geq 2g - 1.$$

$\square$

8. Consequences and the genus-three case

For every $g \geq 2$ and every non-trivial finite character ψ, Theorem 1.1 produces a rank $2g-1$ non-semisimple representation with infinite image and finite ordinary mapping-class-group orbit. Theorem 1.2 says that this rank cannot be improved inside the scalar two-block triangular Prym category.

When $g = 3$, one has

$$\dim V_\psi = 4, \qquad 2g-1 = 5.$$

Thus rank 5 examples exist. A rank 4 scalar two-block compression would have $m + n \leq 4$. Full support would give an injection

$$V_\psi^\vee \hookrightarrow \mathrm{Hom}(N, M),$$

so $4 \leq mn$. The only positive integers with $m+n \leq 4$ and $mn \geq 4$ are $m = n = 2$. Hence a rank 4 scalar two-block compression would identify $V_\psi^\vee$ with a four-dimensional space of 2×2 matrices. Pulling back the determinant would give a finite-orbit projective invariant, contradicting the Prym monodromy consequence above. The general proof replaces this determinant line by all determinantal rank strata and works for every genus and every pair (m, n).

Thus the genus-three obstruction proved here is exactly the scalar two-block Prym obstruction. It leaves open examples arising from non-scalar finite-image semisimple blocks, longer filtrations, or other non-Prym mechanisms.

9. Summary of the proof

We close by recalling how the lower bound fits together. The dimension formula $\dim V_\psi = 2g-2$ gives the size of the universal construction: the identity tensor in $V_\psi \otimes V_\psi^\vee$ produces a triangular representation of rank $2g-1$, and its restriction to $\ker \psi$ contains a non-trivial unipotent element. For an arbitrary scalar two-block extension, the ordinary finite-orbit condition gives, after restricting to H_ψ, a finite orbit of the corresponding K-orbit of the extension class $x \in V_\psi \otimes U$. Together with Looijenga's Prym monodromy consequence, this forces the Prym support of x to be all of V_ψ, hence the contraction map

$$T_x : V_\psi^\vee \longrightarrow \mathrm{Hom}(N, M)$$

is injective.

The rank loci of the image of T_x are determinantal projective subvarieties. Their finite-orbit property and the same Prym monodromy consequence imply that all non-zero matrices in $T_x(V_\psi^\vee)$ have one fixed rank. Westwick's fixed-rank theorem then gives

$$2g-2 = \dim T_x(V_\psi^\vee) \leq \dim M + \dim N - 1,$$

and hence $\dim M + \dim N \geq 2g-1$.

Thus the proof uses Looijenga's cyclic Prym image theorem, Borel density for the arithmetic unitary group appearing in the passage to a single character block, and Westwick's fixed-rank matrix-space bound. The remaining steps are the cohomological identifications and linear algebra carried out above.

Appendix A. Action conventions for the contraction map

Let $x \in V \otimes U$, and let

$$T_x : V^\vee \longrightarrow U$$

be the contraction map

$$T_x(\ell) = (\ell \otimes \mathrm{id}_U)(x).$$

If a group element h acts on V on the left, then it acts on $V^\vee$ by the contragredient rule

$$(h\ell)(v) = \ell(h^{-1}v).$$

Writing $x = \sum_i v_i \otimes u_i$, one obtains

$$T_{hx}(\ell) = \sum_i \ell(hv_i)u_i = T_x(h^{-1}\ell).$$

Consequently $Z_r(hx) = hZ_r(x)$ for the rank loci used in Lemma 6.3. If one adopts the opposite left/right convention, all occurrences of h in this formula are replaced by h^{-1}; the orbit and finite-orbit arguments are unchanged.

Appendix B. Looijenga's notation and the Prym block

For reference we spell out the passage from Looijenga's cyclic theorem to Proposition 3.5. The notation S_C, $S_C^{\#}$, H_C, R_C, and $U^{\#}(H_C)$ is Looijenga's notation from Section 2 of [3], except that $S_C^{\#}$ abbreviates his lifted group $\mathcal{S}^{\#}_{;C}$.

(i) **Cyclic quotient.** Given ψ, choose the finite cyclic quotient $q : H_1(\Sigma_g;\mathbb{Z}) \twoheadrightarrow C$ such that

$$q^*(\widehat{C}) = \langle\psi\rangle \subset \mathrm{Hom}(H_1(\Sigma_g;\mathbb{Z}), \mathbb{C}^\times).$$

Looijenga's subgroup S_C is the subgroup of mapping classes inducing the identity on C, equivalently fixing $\widehat{C}$ pointwise under the dual action. Since $q^*(\widehat{C})$ is generated by ψ, fixing ψ is equivalent to fixing every power of ψ, hence to fixing $q^*(\widehat{C})$ pointwise. Thus S_C is the group denoted here by H_ψ. This is stronger than merely preserving the kernel of q.

(ii) **Lifted group.** Looijenga's Prym representation is naturally defined for a lifted group $S_C^{\#}$ fitting into the central extension

$$1 \longrightarrow C \longrightarrow S_C^{\#} \longrightarrow S_C \longrightarrow 1.$$

The kernel is the deck group. On the fixed ψ-eigenspace, a deck transformation $c \in C$ acts by the scalar $\psi(c)$ or by $\psi(c)^{-1}$, depending on left/right module convention. In either case the action is scalar, hence it maps trivially to $\mathrm{PGL}(E_\psi)$. Therefore $S_C^{\#}$ and $S_C = H_\psi$ have the same connected projective Zariski closure on $\mathbb{P}(E_\psi)$.

(iii) **Prym module.** Looijenga's Proposition 2.2 gives that H_C is a free R_C-module of rank $2(g-1)$ with its skew-Hermitian intersection pairing. After scalar extension $R_C \to \mathbb{C}$ determined by ψ, the block

$$E_\psi = H_C \otimes_{R_C,\psi} \mathbb{C}$$

identifies, with our convention, with the twisted homology block

$$H_1(\Sigma_g; \mathbb{C}_{\psi^{-1}}),$$

and hence

$$E_\psi^\vee \cong H^1(\Sigma_g; \mathbb{C}_\psi) = V_\psi.$$

Under the opposite convention, replace ψ by ψ^{-1}; this does not affect the final projective closure conclusion.

(iv) **Image theorem.** Looijenga's Theorem 2.4 identifies the image of $S_C^{\#}$ on H_C with $U^{\#}(H_C)$ in the cyclic case. This is the Prym monodromy theorem used in the proof.

(v) **Classical group after scalar extension.** Write $K = K_C$ and $K^+ = K_C^0$. Under Looijenga's identification of the image with the arithmetic group $U^{\#}(H_C)$, the connected complex Zariski closure is obtained by base-changing the corresponding K^+-algebraic special unitary group, using Lemma 3.3. For the projective closure considered here, Lemma 3.3 identifies the identity component. As in Lemma 3.4, the finite $C^{(2)}$, determinant, and root-of-unity cosets in Looijenga's $U^{\#}(H_C)$ do not affect that identity component.

If $\psi^2 = 1$, then since $\psi \neq 1$, the character has order two. The block E_ψ is self-dual and Looijenga's skew-Hermitian form becomes a nondegenerate alternating complex bilinear form on E_ψ. The connected complex group on the block is therefore symplectic, not orthogonal, and the connected projective group is $\mathrm{PSp}(E_\psi)$.

If $\psi^2 \neq 1$, then after base change through the real embedding of K^+ determined by $\psi|_{K^+}$,

$$H_C \otimes_{K^+} \mathbb{C} = E_\psi \oplus E_{\psi^{-1}}.$$

The skew-Hermitian form pairs E_ψ perfectly with $E_{\psi^{-1}}$, and each summand is isotropic. The complexified special unitary group acts as

$$A \oplus (A^{-1})^\vee, \qquad A \in \mathrm{SL}(E_\psi).$$

Thus the connected projective group on E_ψ is $\mathrm{PSL}(E_\psi)$. Any determinant, norm-one, or root-of-unity condition left in $U^{\#}(H_C)$ changes at most the finite component group; it cannot cut down the connected $\mathrm{SL}(E_\psi)$ factor after complex base change.

(vi) **Finite-index subgroups.** If $H_0 \leq H_\psi$ is a finite-index subgroup, the image of H_ψ is a finite union of translates of the image of H_0. Their Zariski closures therefore have the same identity component.

The argument above concerns a single character block; simultaneous independence statements for several eigenspaces are not needed.

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