An Endpoint Observation and a Petri-Type Bottleneck for Landesman–Litt Isomonodromy

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Abstract

This note records an endpoint observation and a conditional next-rank refinement for the low-rank semistability analysis of Landesman–Litt in the closed unpointed case. Let

$$s = \left\lceil 2\sqrt{g+1} \right\rceil .$$

Using the unrounded inequality in the proof of the non-generically-globally-generated lemma of Landesman–Litt, together with ordinary degree integrality, we obtain an endpoint sharpening: in the closed unpointed case, an irreducible flat bundle of rank at most s has semistable analytically general isomonodromic deformation. We then analyze the first remaining rank $s+1$. Any rank $s+1$ nonsemistable example must have two Harder–Narasimhan blocks of degrees $(1, -1)$. The Landesman–Litt obstruction becomes a trace-Petri failure, forcing a Hom-jumping condition $\mathrm{Hom}(A, B(p)) \neq 0$. A Segre-strata dimension count, using the standard estimates of Russo–Teixidor–Lange, gives codimension greater than $3g-3$ for the corresponding Hom-jumping envelope. Consequently, rank $s+1$ would be excluded provided one has a suitable isomonodromy–Brill–Noether no-leaf-containment, or dimension-theoretic transversality, statement. We isolate this as the remaining conjectural input. The rank $s+1$ conclusion is therefore conditional, and the note does not construct an example in that rank.

Contents

1 Introduction

Landesman and Litt prove strong restrictions on low-rank flat bundles whose isomonodromic deformations over very general curves fail to be semistable [1]. The numerical threshold relevant here is governed by the inequality

$$r < 2\sqrt{g+1}.$$

The purpose of this note is not to reprove the full Landesman–Litt theorem, nor to address the pointed/parabolic case. We only discuss the closed unpointed case $D = \varnothing$. In that setting, ordinary degree integrality allows one to extract a small endpoint improvement from the proof.

Set

$$N = g+1, \qquad s = \left\lceil 2\sqrt{N} \right\rceil.$$

The first result is an endpoint sharpening of the semistability input.

Theorem 1.1 (Endpoint sharpening; unpointed case). *Let C be a smooth projective curve of genus $g \geq 2$, and let (E, ∇) be an irreducible flat bundle in the closed unpointed setting. If*

$$\operatorname{rk} E \leq s = \left\lceil 2\sqrt{g+1} \right\rceil,$$

then the analytically general isomonodromic deformation of (E, ∇) has semistable underlying vector bundle.

This improves the integer endpoint of the Landesman–Litt semistability bound in this restricted setting. It does not extend, as stated, to the pointed/parabolic case, where parabolic degrees need not be integral.

The next rank is more subtle. Put $n = s+1$. If a nonsemistable example exists in rank n, then its Harder–Narasimhan filtration is forced into a very special form.

Theorem 1.2 (Rank $s+1$ reduction). *In the closed unpointed case, suppose a rank $n = s+1$ irreducible flat bundle has nonsemistable analytically general isomonodromic deformation. Then its Harder–Narasimhan filtration has two blocks*

$$0 \subset A \subset E', \qquad B = E'/A,$$

with

$$\deg A = 1, \qquad \deg B = -1.$$

Thus all remaining numerical candidates are two-block, degree-one profiles. Write

$$a = \operatorname{rk} A, \qquad b = \operatorname{rk} B, \qquad a+b = s+1, \qquad R = ab.$$

They must belong to the finite set

$$\mathcal{Q}_g = \{(a,b) : 1 \leq a \leq b,\ a+b = s+1,\ R(2R - 2g - (s+1)) \geq s+1\}.$$

The set $\mathcal{Q}_g$ is nonempty for all $g \geq 2$; in fact the balanced pair always lies in it. Therefore the refined numerical method alone cannot rule out rank $s+1$. This is not a construction of a rank $s+1$ example.

The remaining obstruction has a Petri-theoretic form. In the two-block case, the second fundamental form is

$$\beta : T_C \longrightarrow A^\vee \otimes B,$$

or equivalently

$$\beta \in H^0(C, A^\vee \otimes B \otimes K_C).$$

Persistence of the HN filtration along the universal first-order isomonodromic deformation gives that

$$H^1(C, T_C) \xrightarrow{H^1(\beta)} H^1(C, A^\vee \otimes B)$$

is the zero map. By Serre duality, this is equivalent to the trace-Petri failure

$$\mathrm{tr}(\gamma \circ \beta) = 0 \qquad \text{for all } \gamma \in H^0(C, B^\vee \otimes A \otimes K_C).$$

This failure forces a Hom-jumping condition $\mathrm{Hom}(A, B(p)) \neq 0$ at the points where $\beta(p) \neq 0$. We prove a codimension estimate for the corresponding Hom-jumping envelope.

Proposition 1.3 (Hom-jumping codimension)**.** *Let $(a, b) \in \mathcal{Q}_g$. For fixed smooth C, define*

$$\mathcal{Z}_{a,b} = \{(A, B, p) : A \in U_C(a, 1),\ B \in U_C(b, -1),\ p \in C, \mathrm{Hom}(A, B(p)) \neq 0\}.$$

Using the standard Segre-strata dimension estimates in the form of Russo–Teixidor–Lange, one has

$$\mathrm{codim}_{U_C(a,1)\times U_C(b,-1)\times C}\, \mathcal{Z}_{a,b} > 3g - 3.$$

More precisely,

$$\mathrm{codim}\, \mathcal{Z}_{a,b} \geq (2g-3)(s+1) - 4g + 8 > 3g - 3.$$

This gives a serious new geometric input, but it still does not yield an unconditional rank $s+1$ exclusion. A high-codimension locus can contain an entire leaf of a foliation. Thus we isolate the missing assumption.

Conjecture 1.4 (Isomonodromy–Brill–Noether no-leaf-containment)**.** *No irreducible rank $s+1$ isomonodromy leaf is locally contained in the relative Petri-bad/Hom-jumping envelope associated with a two-block degree-one profile $(a, b) \in \mathcal{Q}_g$.*

Under this conjecture, or under a stronger dimension-theoretic transversality statement, one obtains a conditional improvement by one further rank.

Theorem 1.5 (Conditional rank $s+1$ exclusion)**.** *Assume either Conjecture 1.4, or the stronger dimension-theoretic transversality statement in Assumption 7.1. Then in the closed unpointed case,*

$$\mathrm{rk}\, E \leq s+1 \quad \Longrightarrow \quad \textit{the analytically general isomonodromic deformation is semistable.}$$

The note is organized as follows. Section 2 records the Landesman–Litt inputs used here. Section 3 proves the endpoint sharpening. Section 4 proves the rank $s+1$ reduction. Sections 5 and 6 give the Petri reformulation and Hom-jumping codimension estimate. Section 7 states the conditional theorem and isolates the remaining transversality problem. The final section summarizes the scope of the results and the external inputs used.

2 Landesman–Litt input

We use the following consequences of the Landesman–Litt analysis. All statements in this section are to be read in the closed unpointed case $D = \varnothing$, although their paper treats a more general logarithmic/parabolic setting.

Proposition 2.1 (Crossing-pair inequality, after Landesman–Litt)**.** *Let (E, ∇) be an irreducible flat bundle, and suppose that an analytically general isomonodromic deformation has nonsemistable underlying vector bundle E'. Let*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_m = E'$$

be the HN filtration and set $G_i = E_i/E_{i-1}$. For every cut $0 < i < m$, there exist indices crossing the cut, say $p \leq i < q$, such that

$$\operatorname{rk}(G_p)\operatorname{rk}(G_q) \geq g + 1.$$

Moreover, for the same selected pair appearing in the proof, the associated slope gap

$$\lambda = \mu(G_p) - \mu(G_q)$$

satisfies

$$0 < \lambda \leq 1.$$

Remark 2.2. The last assertion is included because, in the endpoint argument, the slope bound must apply to the same crossing pair whose Serre-dual bundle is non-generically globally generated. In the proof of Landesman–Litt's Theorem 6.1.1 [1, §6.4.9], this is the pair selected by Lemmas 6.4.3, 6.4.7 and 6.4.8 and then used in Proposition 6.3.6; it should not be replaced by an unrelated adjacent-slope bound.

The second input is the non-generically-globally-generated estimate, but used in an unrounded form.

Lemma 2.3 (Refined non-GGG inequality)**.** *Let V be a semistable vector bundle on C, not generically globally generated, and suppose*

$$\mu(V) = 2g - 2 + \lambda, \qquad 0 < \lambda \leq 1.$$

Then

$$\operatorname{rk} V \geq \frac{2g + \lambda}{2 - \lambda}.$$

Proof. This is extracted from the displayed inequality in the proof of Landesman–Litt's Lemma 6.2.3 [1, Lemma 6.2.3]. Let $U \subset V$ be the saturation of the image of the evaluation map. Since V is not generically globally generated, $U \subsetneq V$, so

$$c = \operatorname{rk} V - \operatorname{rk} U \geq 1.$$

Moreover $H^0(U) = H^0(V)$, hence the defect term

$$\delta = h^0(V) - h^0(U)$$

is zero. The proof of the Landesman–Litt lemma gives

$$(\mu(V)+2)c \le (2g-\mu(V))\,\mathrm{rk}\,V + 2\delta.$$

Substituting $\delta = 0$ and $\mu(V) = 2g-2+\lambda$, we get

$$(2g+\lambda)c \le (2-\lambda)\,\mathrm{rk}\,V.$$

Since $c \ge 1$, the result follows. $\square$

Remark 2.4. In the Landesman–Litt proof, the relevant bundle is of the form

$$V = \mathrm{Hom}(G_p, G_q)^\vee \otimes K_C.$$

The HN graded pieces are semistable, and over a characteristic-zero curve the dual/tensor operations preserve the semistability needed here in this setup.

3 Endpoint sharpening

We now prove Theorem 1.1.

Proof of Theorem 1.1. Let $r = \mathrm{rk}\,E$. If $r < s$, the Landesman–Litt crossing-pair inequality immediately gives a contradiction: a pair of positive integers x, y with $xy \ge g+1 = N$ satisfies

$$x + y \ge 2\sqrt{N},$$

hence, by integrality,

$$x + y \ge s.$$

But $x + y \le r < s$.

It remains to exclude $r = s$. Suppose E' is nonsemistable. If the HN filtration had at least three graded pieces, a selected crossing pair would have rank sum at least s, and there would be at least one further positive-rank HN block. Thus the total rank would exceed s, impossible. Hence the HN filtration has two blocks:

$$0 \subset A \subset E', \qquad B = E'/A.$$

Set

$$a = \mathrm{rk}\,A, \quad b = \mathrm{rk}\,B, \quad a + b = s, \quad R = ab.$$

Since $D = \varnothing$, the flat bundle has degree zero. Write

$$\deg A = d > 0, \qquad \deg B = -d.$$

The slope gap is

$$\lambda = \mu(A) - \mu(B) = \frac{d}{a} + \frac{d}{b} = \frac{ds}{R} \ge \frac{s}{R}.$$

By Proposition 2.1 and Lemma 2.3, applied to

$$V = \mathrm{Hom}(A, B)^\vee \otimes K_C,$$

whose rank is R, we have

$$R \geq \frac{2g+\lambda}{2-\lambda}.$$

Since the right-hand side is increasing in λ for $0 < \lambda < 2$,

$$R \geq \frac{2g+s/R}{2-s/R}.$$

Equivalently,

$$R(2R-2g-s) \geq s. \tag{$*$}$$

But $a+b=s$, hence

$$R = ab \leq \left\lfloor \frac{s^2}{4} \right\rfloor.$$

We now derive a contradiction by parity.

If $s = 2k$, then $s = \lceil 2\sqrt{N} \rceil = 2k$ implies

$$(k-1/2)^2 < N \leq k^2.$$

Since N is integral,

$$N \geq k^2 - k + 1.$$

Thus

$$R \leq k^2,$$

and

$$2R - 2g - s \leq 2k^2 - 2(N-1) - 2k = 2(k^2 - k + 1 - N) \leq 0.$$

So the left-hand side of $(*)$ is nonpositive, contradiction.

If $s = 2k+1$, then

$$k^2 < N \leq (k+1/2)^2,$$

so $N \geq k^2 + 1$. Since $R \leq k(k+1)$,

$$2R - 2g - s \leq 2k(k+1) - 2(N-1) - (2k+1) = 2k^2 - 2N + 1 < 0.$$

Again $(*)$ is impossible. This proves the endpoint result. □

4 The first remaining rank

Set

$$n = s + 1.$$

We prove Theorem 1.2 in several steps.

Lemma 4.1 (Arithmetic lemma). *Let $N \geq 4$ be an integer and set $s = \lceil 2\sqrt{N} \rceil$. If $x, c, y \in \mathbb{Z}_{>0}$ satisfy*

$$cx \geq N, \qquad cy \geq N,$$

then

$$x + c + y \geq s + 2.$$

Proof. Suppose, for contradiction, that $x + c + y \le s + 1 =: M$. Then

$$\min(cx, cy) \le c\frac{x+y}{2} \le c\frac{M-c}{2} \le \frac{M^2}{8}.$$

For $N \ge 6$, since $M = s + 1 \le 2\sqrt{N} + 2$, one checks

$$\frac{M^2}{8} < N.$$

Thus $\min(cx, cy) < N$, contradicting the hypotheses. The remaining cases $N = 4, 5$ are checked directly. □

Proposition 4.2 (Rank $s+1$ HN length)**.** *Let $r = s + 1$, and suppose E' is nonsemistable. Then the HN filtration of E' has length two.*

Proof. If the HN length were at least four, then for the first cut Proposition 2.1 would produce a crossing pair with rank sum at least s, and at least two further positive-rank HN blocks would remain. Hence the total rank would be at least $s + 2$, contradiction.

Suppose the HN length is three, with slopes

$$\mu_1 > \mu_2 > \mu_3.$$

If $\mu_2 > 0$, then $\mu_3 < 0$, and the second cut yields a positive/negative crossing pair. Its rank sum is at most s. If the sum is less than s, AM–GM contradicts the product bound. If the sum is s, the endpoint argument above applies to this positive/negative pair: both degrees are nonzero with opposite signs, so the slope gap satisfies $\lambda \ge s/R$, and the same refined inequality gives the contradiction. The case $\mu_2 < 0$ is analogous.

Thus the only remaining possibility is

$$\mu_1 > 0, \qquad \mu_2 = 0, \qquad \mu_3 < 0.$$

Let the ranks be x, c, y. To avoid a positive/negative selected pair, the two cuts must produce the products

$$cx \ge N, \qquad cy \ge N.$$

For $g \ge 3$, Lemma 4.1 gives

$$x + c + y \ge s + 2,$$

contradicting $r = s + 1$. For $g = 2$, we have $N = 3$, $s = 4$, $r = 5$, and the only possible pattern is $(x, c, y) = (1, 3, 1)$. The corresponding rank product is $R = 3$ and the slope gap is $\lambda = 1$; Lemma 2.3 would require

$$3 \ge \frac{2g+1}{1} = 5,$$

which is impossible. Hence length three is also excluded. □

Proposition 4.3 (Degree split in rank $s+1$)**.** *If a rank $s+1$ nonsemistable example exists, then its two HN blocks have degrees $(1, -1)$.*

Proof. By Proposition 4.2, the HN filtration has two blocks

$$0 \subset A \subset E', \qquad B = E'/A.$$

Let

$$a + b = n = s + 1, \qquad R = ab,$$

and write

$$\deg A = d > 0, \qquad \deg B = -d.$$

The slope gap is

$$\lambda = \frac{dn}{R}.$$

Applying Lemma 2.3 gives

$$R \geq \frac{2g + dn/R}{2 - dn/R},$$

equivalently

$$R(2R - 2g - dn) \geq dn. \tag{$**$}$$

We show that $d \geq 2$ is impossible. It suffices to check $d = 2$, since larger d only strengthens the obstruction. If $s = 2k$, then $n = 2k + 1$, $R \leq k(k + 1)$, and one may write

$$N = g + 1 = k^2 - u, \qquad 0 \leq u \leq k - 1.$$

At the maximal value of R,

$$2R - 2g - 2n \leq 2u - 2k < 0.$$

Thus ($**$) cannot hold. If $s = 2k + 1$, then $n = 2k + 2$, $R \leq (k + 1)^2$, and

$$N = g + 1 = k^2 + 1 + u, \qquad 0 \leq u \leq k - 1.$$

At the maximal value of R,

$$2R - 2g - 2n \leq -2u - 2 < 0.$$

Again ($**$) is impossible. Hence $d = 1$. □

Proof of Theorem 1.2. Combine Propositions 4.2 and 4.3. □

Definition 4.4 (Surviving numerical profiles)**.** Let $\mathcal{Q}_g$ be the set of pairs (a, b) satisfying

$$1 \leq a \leq b, \qquad a + b = s + 1, \qquad R = ab, \qquad R(2R - 2g - (s + 1)) \geq s + 1.$$

Proposition 4.5 (Non-emptiness of $\mathcal{Q}_g$)**.** *For every $g \geq 2$, the balanced pair*

$$\left(\left\lfloor \frac{s+1}{2} \right\rfloor, \left\lceil \frac{s+1}{2} \right\rceil \right)$$

belongs to $\mathcal{Q}_g$. In particular, $\mathcal{Q}_g \neq \varnothing$.

Proof. If $s = 2k$, then $s+1 = 2k+1$, the balanced pair is $(k, k+1)$, and $R = k(k+1)$. Since $s = \lceil 2\sqrt{N} \rceil = 2k$, we have $N \leq k^2$. Thus

$$2R - 2g - (s+1) = 2k(k+1) - 2(N-1) - (2k+1) = 2k^2 - 2N + 1 \geq 1.$$

Therefore

$$R(2R - 2g - (s+1)) \geq R \geq s+1.$$

If $s = 2k+1$, then $s+1 = 2k+2$, the balanced pair is $(k+1, k+1)$, and $R = (k+1)^2$. Now $N \leq k^2 + k$, hence

$$2R - 2g - (s+1) = 2(k+1)^2 - 2(N-1) - (2k+2) = 2(k^2 + k + 1 - N) \geq 2.$$

Again

$$R(2R - 2g - (s+1)) \geq 2R \geq s+1.$$

□

Remark 4.6. Proposition 4.5 says only that the refined numerical inequalities cannot rule out rank $s+1$. It does not construct a rank $s+1$ counterexample.

5 Petri reformulation

Assume now that we are in the remaining rank $s+1$ two-block degree-one situation. Thus

$$0 \subset A \subset E', \qquad B = E'/A,$$

with

$$\deg A = 1, \qquad \deg B = -1.$$

The second fundamental form of the connection is a map

$$\beta : T_C \longrightarrow A^\vee \otimes B,$$

equivalently a section

$$\beta \in H^0(C, A^\vee \otimes B \otimes K_C).$$

Proposition 5.1 (Trace-Petri form of the obstruction)**.** *In the two-block case, the Landesman–Litt first-order persistence obstruction is the vanishing of*

$$H^1(C, T_C) \xrightarrow{H^1(\beta)} H^1(C, A^\vee \otimes B).$$

By Serre duality this is equivalent to

$$\operatorname{tr}(\gamma \circ \beta) = 0 \quad \text{for all } \gamma \in H^0(C, B^\vee \otimes A \otimes K_C).$$

Proof. Since $T_C^\vee = K_C$, a morphism $T_C \to A^\vee \otimes B$ is the same as a section of $A^\vee \otimes B \otimes K_C$. The induced cohomology map is

$$H^1(T_C) \to H^1(A^\vee B).$$

Serre duality identifies

$$H^1(T_C)^\vee \simeq H^0(K_C^2)$$

and

$$H^1(A^\vee B)^\vee \simeq H^0((A^\vee B)^\vee \otimes K_C) = H^0(B^\vee A K_C).$$

The dual map sends a section $\gamma \in H^0(B^\vee A K_C)$ to the quadratic differential

$$\mathrm{tr}(\gamma \circ \beta) \in H^0(K_C^2).$$

Thus the original map is zero precisely when all these trace products vanish. □

6 The Hom-jumping envelope

Let

$$G = B^\vee \otimes A \otimes K_C.$$

The following elementary observation turns trace-Petri failure into a Hom-jumping condition.

Lemma 6.1 (Petri-bad implies Hom-jumping)**.** *Suppose* $0 \neq \beta \in H^0(A^\vee B K_C)$ *satisfies the trace-Petri failure of Proposition 5.1. If* $p \in C$ *and* $\beta(p) \neq 0$*, then*

$$\mathrm{Hom}(A, B(p)) \neq 0.$$

Proof. If G were generated at p, then the fiberwise trace pairing

$$\mathrm{Hom}(A_p, B_p) \times \mathrm{Hom}(B_p, A_p) \to \mathbb{C}$$

would allow us to choose a value $\gamma(p) \in G_p$ such that

$$\mathrm{tr}(\gamma(p) \circ \beta(p)) \neq 0.$$

Generation at p lifts this value to a global section $\gamma \in H^0(G)$, contradicting trace-Petri failure. Hence G is not generated at p.

Since

$$H^1(G)^\vee = H^0(A^\vee B) = \mathrm{Hom}(A, B) = 0$$

by stability and $\mu(A) > \mu(B)$, the exact sequence

$$0 \to G(-p) \to G \to G_p \to 0$$

shows that non-generation at p implies $H^1(G(-p)) \neq 0$. By Serre duality,

$$H^1(G(-p))^\vee = H^0(A^\vee B(p)) = \mathrm{Hom}(A, B(p)).$$

Thus $\mathrm{Hom}(A, B(p)) \neq 0$. □

Definition 6.2 (Hom-jumping locus)**.** For $(a, b) \in \mathcal{Q}_g$, define

$$\mathcal{Z}_{a,b} = \{(A, B, p) : A \in U_C(a, 1),\ B \in U_C(b, -1),\ p \in C, \mathrm{Hom}(A, B(p)) \neq 0\}.$$

We now prove Proposition 1.3. The proof is a dimension estimate. We use the standard Segre-strata and twisted Brill–Noether dimension estimates for stable bundles developed by Russo–Teixidor [2]: in particular, the dimension formula for the strata $U_{r',s}(r, d)$ stated in their introduction and their twisted Brill–Noether theorem for one section. These estimates are used only to justify the expected dimensions of the subbundle and map strata below; the remaining inequalities are elementary.

Proof of Proposition 1.3. The ambient coarse moduli dimension is

$$\dim U_C(a,1) + \dim U_C(b,-1) + 1 = (a^2+b^2)(g-1) + 3.$$

Equivalently, on stable stacks the ambient dimension differs by a constant gerbe dimension, so codimensions are the same.

Fix $p \in C$, and set

$$E = B(p).$$

Then E is stable of rank b and degree $b-1$. Consider a nonzero map

$$f : A \to E.$$

Let $J = \operatorname{im}(f)$, let

$$k = \operatorname{rk} J, \qquad d = \deg J,$$

and let $I = J^{\mathrm{sat}} \subset E$ have degree e. Put $t = e - d \geq 0$.

First, $k = 1$ is impossible. If $k < a$, stability of A gives, for the quotient $A \twoheadrightarrow J$,

$$\mu(J) > \mu(A) = \frac{1}{a}.$$

For $k = 1$, this forces $d \geq 1$. But $I \subset E$ is a line subbundle of a stable bundle with slope

$$\mu(E) = \frac{b-1}{b} < 1,$$

so $e \leq 0$. Since $d \leq e$, this is impossible.

Now assume $2 \leq k < a$. Again $d \geq 1$. In the saturated case $I = J$, we have exact sequences

$$0 \to K \to A \to J \to 0,$$

and

$$0 \to J \to E \to Q \to 0,$$

where

$$\operatorname{rk} K = a - k, \quad \deg K = 1 - d,$$

$$\operatorname{rk} Q = b - k, \quad \deg Q = b - 1 - d.$$

For vector-bundle stacks, $\mathrm{Bun}_{r,d}$ has dimension $r^2(g-1)$, and the exact-sequence stack

$$0 \to X \to Z \to Y \to 0$$

has relative dimension $-\chi(Y^\vee \otimes X)$ over $\mathrm{Bun}_X \times \mathrm{Bun}_Y$. Expanding the two extension counts and comparing with the ambient dimension yields the codimension lower bound

$$c_{k,d} = (b-a)d + k\big(a(g-1) + b(g-2) + 2\big) - k^2(g-1).$$

If $t = e - d > 0$, the same calculation gives

$$c_{k,d,t} = c_{k,d} + t(b-k),$$

so the worst case is saturated.

Since $b \geq a$, the expression is minimized in degree at $d = 1$. Put $c_k = c_{k,1}$. The function c_k is a concave quadratic in k, so on $2 \leq k \leq a-1$ its minimum occurs at an endpoint. A direct calculation gives

$$c_2 = (2g-3)(a+b) - 4g + 8,$$

and

$$c_{a-1} - c_2 = (a-3)(bg - 2b - g + 3) \geq 0$$

for $a \geq 3$, $g \geq 2$. If $a = 2$, there is no nonmaximal rank stratum. Hence all nonmaximal rank strata have codimension at least c_2.

It remains to treat the maximal-rank case $k = a$. If $a < b$, then a map $A \to E$ is generically injective. If the saturation has torsion length t, an elementary modification contributes at most at parameters, and the quotient has rank $b - a$. The extension-stack count gives

$$c_{\max,t} = ab(g-2) + a + b + t(b-a),$$

so the worst case is

$$c_{\max} = ab(g-2) + a + b.$$

If $a = b$, the quotient is torsion of length $a - 2$; the elementary transformation count gives the same lower bound

$$c_{\max} = a^2(g-2) + 2a.$$

Finally,

$$c_{\max} - c_2 = (a-2)(b-2)(g-2) \geq 0.$$

Thus every stratum has codimension at least

$$c_2 = (2g-3)(a+b) - 4g + 8.$$

Since $a + b = s + 1$,

$$\operatorname{codim} \mathcal{Z}_{a,b} \geq (2g-3)(s+1) - 4g + 8.$$

For $g = 2$, $s = 4$ and this gives $5 > 3 = 3g - 3$. For $g = 3$, it gives $11 > 6 = 3g - 3$. For $g \geq 4$, we have $s + 1 \geq 6$, hence

$$(2g-3)(s+1) - 4g + 8 \geq 6(2g-3) - 4g + 8 = 8g - 10 > 3g - 3.$$

This proves the claimed estimate. □

Remark 6.3. The only external input in Proposition 1.3 is the standard Segre-strata/twisted Brill–Noether dimension theory cited above. The stack/coarse comparison is harmless for stable bundles, because the stable stack is a constant $\mathbb{G}_m$-gerbe over the coarse moduli space.

7 Conditional rank $s+1$ exclusion

Let $\mathcal{W}_{a,b}$ denote the relative Petri-bad/Hom-jumping envelope over the relative de Rham moduli, associated with a fixed surviving profile $(a,b) \in \mathcal{Q}_g$. Proposition 1.3 gives a vertical codimension estimate for the Hom-jumping envelope. To use it against isomonodromy leaves, one needs an additional foliation-theoretic input.

Assumption 7.1 (No-leaf-containment / dimension-theoretic transversality)**.** *For every irreducible rank $s+1$ representation ρ, the isomonodromy leaf $\mathcal{L}_\rho$ is not locally contained in $\mathcal{W}_{a,b}$. Moreover, at every component of $\mathcal{L}_\rho \cap \mathcal{W}_{a,b}$, the codimension inside the leaf is at least the ambient codimension of $\mathcal{W}_{a,b}$. Thus, in the present Petri/Hom-jumping setting, a locus of ambient codimension greater than $3g-3$ cannot contain a positive-dimensional open subset of the $3g-3$-dimensional isomonodromy leaf.*

Proof of Theorem 1.5. Assume a rank $s+1$ nonsemistable analytically general isomonodromic deformation exists. By Theorem 1.2, its HN type is two-block degree-one. Landesman–Litt's Shatz-stratification argument implies that, after replacing the base by an analytic open subset of the isomonodromy leaf, this HN filtration persists. Hence, by Proposition 5.1 and Lemma 6.1, an open subset of the isomonodromy leaf lies in the relative Petri-bad/Hom-jumping envelope $\mathcal{W}_{a,b}$.

If Conjecture 1.4 is assumed, this already contradicts no-leaf-containment. Under the stronger Assumption 7.1, Proposition 1.3 gives ambient codimension greater than $3g-3$. The assumed dimension-theoretic transversality would force the same lower bound for the codimension inside the $3g-3$-dimensional leaf, which is impossible for a subset containing an open piece of the leaf. This is again a contradiction. Hence rank $s+1$ nonsemistability is excluded under either stated hypothesis. □

8 The remaining transversality problem

The conditional nature of Theorem 1.5 is essential. High codimension alone does not imply avoidance by a leaf of a foliation. For example, in a product $X = Y \times L$ foliated by the copies $\{y\} \times L$, the subvariety $\{y_0\} \times L$ can have codimension $\dim Y$, larger than $\dim L$, and still contain a whole leaf.

Thus the missing input is not a further Brill–Noether dimension estimate, but a no-leaf-containment or transversality statement for the isomonodromy foliation against the Petri-bad/Hom-jumping strata. The local deformation picture is as follows. The Atiyah sequence [3]

$$0 \to \operatorname{End} E \to \operatorname{At}(E) \to T_C \to 0$$

and its filtered quotient

$$0 \to \operatorname{At}(E, A) \to \operatorname{At}(E) \to A^\vee B \to 0$$

show that the connection produces

$$\beta : T_C \to A^\vee B.$$

The condition $H^1(\beta) = 0$ says precisely that the isomonodromy directions are tangent to the HN incidence. It does not, by itself, compute the normal derivative to the Hom-jumping locus $\operatorname{Hom}(A, B(p)) \neq 0$. That latter locus is controlled by deformations of a nonzero map $A \to B(p)$, whose obstruction lives in $H^1(A^\vee B(p))$. Relating these two deformation complexes is the missing step.

Known constructions also warn against assuming a general avoidance theorem: Landesman–Litt construct high-rank examples whose isomonodromic deformations are never semistable. Thus any transversality theorem must use the low-rank, two-block, degree-one structure in an essential way.

Conjecture 8.1 (No-leaf-containment, precise form)**.** *Let $(a, b) \in \mathcal{Q}_g$. No irreducible rank $s+1$ isomonodromy leaf is locally contained in the relative Petri-bad locus associated with the two-block degree-one HN profile (a, b).*

This conjecture is the remaining point needed for the conditional route described above.

9 Scope and external inputs

The unconditional part of the note is limited to the closed unpointed case and consists of Theorem 1.1 and the rank $s+1$ reduction in Theorem 1.2. The latter is a reduction to two Harder–Narasimhan blocks of degrees $(1,-1)$, not a construction of a rank $s+1$ example and not an unconditional exclusion in that rank.

The conditional part is Theorem 1.5. It uses the Petri/Hom-jumping codimension estimate together with an additional no-leaf-containment or dimension-theoretic transversality hypothesis for isomonodromy leaves. High codimension of the Hom-jumping envelope alone is not enough to imply avoidance by a leaf of a foliation.

The external inputs used in the paper are the following. The endpoint and rank $s+1$ reductions use the crossing-pair mechanism and non-generically-globally-generated estimates from Landesman–Litt, as recorded in Section 2. The Hom-jumping codimension estimate uses standard Segre-strata and twisted Brill–Noether dimension estimates in the form developed by Russo–Teixidor. The remaining arguments are elementary arithmetic, Serre duality, and dimension counting for exact-sequence stacks.

References

[1] Aaron Landesman and Daniel Litt, *Geometric local systems on very general curves and isomonodromy*, Journal of the American Mathematical Society **37** (2024), 683–729. See also arXiv:2202.00039.

[2] B. Russo and M. Teixidor i Bigas, *On a conjecture of Lange*, arXiv:alg-geom/9710019.

[3] M. F. Atiyah, *Complex analytic connections in fibre bundles*, Transactions of the American Mathematical Society **85** (1957), 181–207.